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On the extension of lattice-valued implications via retractions

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Abstract

The main goal of this paper is to apply the method of extension of fuzzy connectives proposed in our previous work for fuzzy implications valued on a bounded lattice. Also we discuss about which properties of implications are preserved by this method and we prove some results involving extension and automorphisms. Finally, we investigate the behavior of the extensions of two special classes of fuzzy implications, namely (S, N)-implications and R-implications.

Keywords: Extension; Lattice-valued implication; Lattice automorphism; Retraction

1. Introduction

A well-known problem which has been studied by many researchers is how to extend a given function from a subset to a larger domain, preserving its main properties; that is, if L is an arbitrary set and if f is a function defined on a nonempty subset M of L which verifies a given property P, how can f be extended to L in order to preserve property P for elements of $L \setminus M$? For instance, an important theorem of analysis states that a continuous function f defined on a bounded closed set M of the real line can be extended to the whole real line preserving its continuity [1,4,22].

In fuzzy logic, the problem of extending functions can be considered for lattice-valued fuzzy connectives (t-norms, t-conorms, negations and others) since these connectives are functions, in particular. The pioneer work in this framework was published by Saminger-Platz et al. in [27] providing a method to extend a t-norm T from a complete sublattice M to a bounded lattice L. In [25] we have developed an alternative method to extend t-norms, t-conorms and fuzzy negations which, by considering a modified notion of sublattice, generalizes the method proposed in [27].

In this paper we apply an extension method proposed in [25] to fuzzy implications. The main goals of this work are: (1) to define a generic method to extend fuzzy implications from an (r, s)-sublattice (see Definition 2.5) and to check which properties are preserved by this extension, and (2) to make a particular study of the behavior of this extension for two special classes of fuzzy implications, namely, (S, N)-implications and R-implications.

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The paper is organized as follows: In Section 2 we discuss some concepts (lattices, sublattices, homomorphisms, retractions) that we use throughout the paper. The extension method proposed in [25] and some new results for t-norms, t-conorms and negations are presented in Section 3. Within the framework of this extension, in Section 4 we apply our method to implications and investigate which properties are preserved. Finally, in Sections 5 and 6, we turn our attention to extensions of (S, N)-implications and R-implications, respectively.

2. Preliminaries

In this section we present a formalization of the main concepts that will be used in this work such as lattice, sublattice, homomorphism, retractions and so on. For further reading about these concepts we recommend [8,12,13, 15,17,19–21,28].

2.1. Bounded lattices: definition and related concepts

It is known from the literature that the concept of lattice can be approached in two ways, i.e. as a poset and as an algebraic structure. Here we just present the algebraic definition of lattice for reasons that will be clear by the context, but a discussion about these approaches can be found in [8,25].

Definition 2.1. Let L be a nonempty set. If $\wedge_L$ and $\vee_L$ are two binary operations on L, then $\langle L, \wedge_L, \vee_L\rangle$ is a *lattice* provided that for each $x, y, z \in L$, the following properties hold:

1. $x \wedge_L y = y \wedge_L x$ and $x \vee_L y = y \vee_L x$ (symmetry);
2. $(x \wedge_L y) \wedge_L z = x \wedge_L (y \wedge_L z)$ and $(x \vee_L y) \vee_L z = x \vee_L (y \vee_L z)$ (associativity);
3. $x \wedge_L (x \vee_L y) = x$ and $x \vee_L (x \wedge_L y) = x$ (absorption laws).

If in $\langle L, \wedge_L, \vee_L\rangle$ there are elements 0_L and 1_L such that, for all $x \in L$, $x \wedge_L 1_L = x$ and $x \vee_L 0_L = x$, then $\langle L, \wedge_L, \vee_L, 0_L, 1_L\rangle$ is called a *bounded lattice*.

Moreover, it is known that, given a lattice L, the relation

$$x \leqslant_L y \quad \text{if and only if} \quad x \wedge_L y = x \tag{1}$$

defines a partial order on L. This order will be used by us to compare elements.

Recall also that a lattice L is called a complete lattice if every subset of L has a supremum and an infimum element.[1]

Example 2.1. The set [0, 1] endowed with the operations defined by $x \wedge y = \min\{x, y\}$ and $x \vee y = \max\{x, y\}$ for all $x, y \in [0, 1]$ is a (complete) bounded lattice in the sense of Definition 2.1 which has 0 as the bottom and 1 as the top element.

Remark 2.1. In order to simplify the notation, throughout this paper when we say that L is a bounded lattice it means that L has a structure as in Definition 2.1. If L denotes a different structure, an appropriate distinction will be made.

Remark 2.2. When $\leqslant_L$ is a partial order on L and there are two elements x and y belonging to L such that neither $x \leqslant_L y$ nor $y \leqslant_L x$, these elements are said to be incomparable and we denote this by $x \parallel y$. Otherwise we say that x and y are comparable (notation: $x \supset\!\!\subset y$).

Definition 2.2. Let $(L, \wedge_L, \vee_L, 0_L, 1_L)$ and $(M, \wedge_M, \vee_M, 0_M, 1_M)$ be bounded lattices. A mapping $f : L \to M$ is said to be a *lattice homomorphism* if, for all $x, y \in L$, we have

1. $f(x \wedge_L y) = f(x) \wedge_M f(y)$;

[1] An element $a \in L$ is called a supremum (resp. infimum) of L if (1) $x \leqslant_L a$ (resp. $x \geqslant_L a$) for all $x \in L$; and (2) if there exists $b \in L$ satisfying (1), then $a <_L b$ (resp. $a >_L b$). Note that, $\inf \emptyset = 1_L$ and $\sup \emptyset = 0_L$.

2. $f(x \vee_L y) = f(x) \vee_M f(y)$;
3. $f(0_L) = 0_M$ and $f(1_L) = 1_M$.

Remark 2.3. Recall that, an injective (a surjective) lattice homomorphism is called a monomorphism (epimorphism) and a bijective lattice homomorphism is called an isomorphism. An automorphism is an isomorphism from a lattice onto itself.

Proposition 2.1. *(See [25, Proposition 2.1].) Every lattice homomorphism preserves the order.*

Proposition 2.2. *(See [24, Proposition 2.2].) Let L be a bounded lattice. Then a function $\rho : L \to L$ is an automorphism if and only if*

1. *ρ is bijective and*
2. *$x \leqslant_L y$ if and only if $\rho(x) \leqslant_L \rho(y)$.*

From now on, lattice homomorphisms will just be called homomorphisms for simplicity.

Remark 2.4. We denote the set of all automorphisms on a bounded lattice L by $Aut(L)$. This set endowed with the function composition operation is a group which has as neutral element the identity function id_L. In algebra, an important tool is the action of the groups on sets [9,18]. In our case the action of the automorphism group transforms lattice functions on other lattice functions.

Definition 2.3. Given a function $f : L^n \to L$, the *action of an L-automorphism ρ* over f is the function $f^\rho : L^n \to L$ defined by

$$f^\rho(x_1, \dots, x_n) = \rho^{-1}\big(f\big(\rho(x_1), \dots, \rho(x_n)\big)\big). \tag{2}$$

In this case, f^ρ is said to be a *conjugate* of f (see [14]).

Notice that if $f : L^n \to L$ is a conjugate of $g : L^n \to L$ and g is a conjugate of the $h : L^n \to L$ then f is a conjugate of h (automorphisms are closed under composition) and if f is a conjugate of g then, since the inverse of an automorphism is also an automorphism, g is a conjugate of f. Thus, an automorphism action on the set of n-ary functions on L (L^{L^n}) determines an equivalence relation on L^{L^n}.

Let $f : L^n \to L$ be a conjugate of $g : L^n \to L$. If $f(x_1, \dots, x_n) \leqslant_L g(x_1, \dots, x_n)$ for each $x_1, \dots, x_n \in L$ then we write $f \leqslant g$.[2]

2.2. Retracts and sublattices

In general, given a bounded lattice L and a nonempty subset $M \subseteq L$, it is said that M is a sublattice of L if, for all $x, y \in M$, the following conditions hold:

$$x \wedge_L y \in M \quad \text{and} \quad x \vee_L y \in M.$$

In other words, M equipped with the restriction of the operations $\wedge_L$ and $\vee_L$ inherits the lattice structure of L.

We would like to work with a relaxed notion of sublattice in which the condition $M \subseteq L$ is somewhat weakened.

Definition 2.4. (See [8].) A homomorphism r of a lattice L onto a lattice M is said to be a retraction if there exists a homomorphism s of M into L which satisfies $r \circ s = id_M$. A lattice M is called a retract of a lattice L if there is a retraction r, of L onto M, and s is then called a pseudo-inverse of r.

It is important to point out here that a retraction r as in Definition 2.4 is surjective (onto) and hence s is naturally injective since $r \circ s = id_M$. These properties are used in some proofs throughout this paper.

[2] Notice that an ordering between functions f and g can be defined as we have made here even when f is not a conjugate of g.

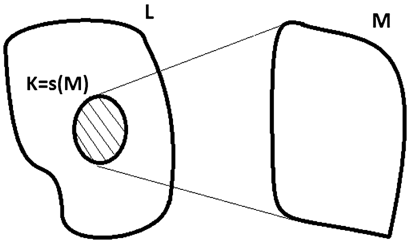

Fig. 1. The relaxed idea of sublattice.

Definition 2.5. Let L and M be arbitrary bounded lattices. We say that M is an (r, s)-sublattice of L if M is a retract of L (i.e. M is a sublattice of L up to isomorphisms). In other words, M is an (r, s)-sublattice of L if there is a retraction r of L onto M with pseudo-inverse $s : M \to L$.

The purpose of Definition 2.5 is to provide a relaxed notion of the concept of sublattice. It is done an identification of M with a subset $K = s(M)$ of L in order to bring some properties from M to K via the retraction r, including the lattice structure (see Fig. 1). In this case, K works as an algebraic copy of M embedded into L since r is a homomorphism.

Notice that every complete sublattice in the usual sense is also an (r, s)-sublattice in our sense. It is enough to consider as s the inclusion of M into L and as r the mapping which keeps unchanged M, sends $x \in L \backslash M$ to the supremum (in M) of all $z \in M$ such that $z \leqslant_L x$ (i.e. $r(x) = \sup_M \{z \in M \mid z \leqslant_L x\}$).

Remark 2.5. Throughout this paper, we consider the notion of (r, s)-sublattice as in Definition 2.5 instead of the ordinary notion of sublattice. Whenever the usual definition of sublattice is used and this is not clear from the context, this sublattice will be called ordinary sublattice.

Definition 2.6. Every retraction $r : L \to M$ (with pseudo-inverse s) which satisfies $s \circ r \leqslant id_L$[3] ($id_L \leqslant s \circ r$) is called a lower (an upper) retraction. In this case, M is called a lower (an upper) retract of L.

Notice that in both Definitions 2.5 and 2.6 the pseudo-inverse s of a retraction r needs not be unique. This is an advantage of our notion of sublattice since if there exist more than one pseudo-inverse for the same retraction it is possible to identify M with a subset of L in different ways. This gives us the possibility of choosing the best one for our proposes. But it must be clear that when we say that M is a (lower, upper or neither) (r, s)-sublattice of L we are considering the existence of at least one pseudo-inverse s and fixing it. In any case, no matter which pseudo-inverse is taken, every result presented here holds.

Example 2.2. One can easily see in Fig. 2 that M is a lower retract (but it is not an upper retract) of L_1, M is an upper retract (but is not a lower retract) of L_2 and M is a retract (but is neither an upper nor a lower retract) of L_3. In fact, the unique possible retracts are $r_i : L_i \to M$ with $i \in \{1, 2, 3\}$ defined as in Table 1.

Their pseudo-inverses $s_i : M \to L_i$ with $i \in \{1, 2, 3\}$ are respectively given as in Table 2.

Example 2.3. Let M and L be bounded lattices as shown in Fig. 3. A mapping $r : L \to M$ given by $r(x) = \sup\{z \in M \mid s(z) \leqslant_L x\}$ is a lower retraction whose pseudo-inverse is the mapping $s : M \to L$ defined by $s(1_M) = 1_L$, $s(a) = v$, $s(b) = x$, $s(c) = y$, $s(d) = z$ and $s(0_M) = 0_L$. Therefore, it follows that M is an (r, s)-sublattice of L in the sense of Definition 2.5.

Remark 2.6. Note that, given a lower retraction, it is sometimes possible to define an upper retraction with the same pseudo-inverse. For instance, let L and M be the lattices shown in Fig. 3. If r is a lower retraction with pseudo-inverse s as defined in Example 2.3, then the function r' given by $r'(x) = \inf\{z \in M \mid s(z) \geqslant_L x\}$ is an upper retraction since $id_L \leqslant s \circ r'$. It is easy to check that s is also a pseudo-inverse of r'.

[3] If f and g are functions on a lattice L it is said that $f \leqslant g$ if and only if $f(x) \leqslant_L g(x)$ for all $x \in L$.

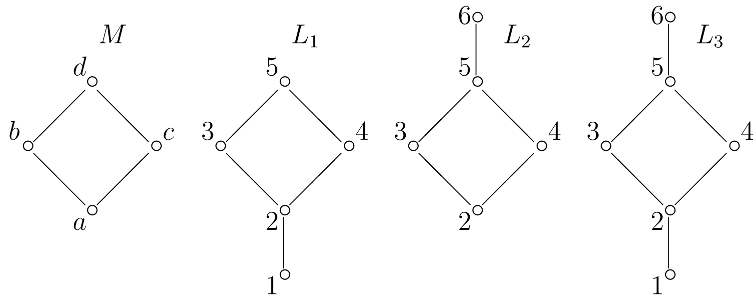

Fig. 2. Hasse diagrams of lattices M, L_1, L_2 and L_3.

Table 1
Tables of retractions r_1, r_2 and r_3.

x	r_1	x	r_2	x	r_3
1	a	2	a	1	a
2	a	3	b	2	a
3	b	4	c	3	b
4	c	5	d	4	c
5	d	6	d	5	d
				6	d

Table 2
Tables of pseudo-inverses s_1, s_2 and s_3.

x	s_1	s_2	s_3
a	1	2	1
b	3	3	3
c	4	4	4
d	5	6	6

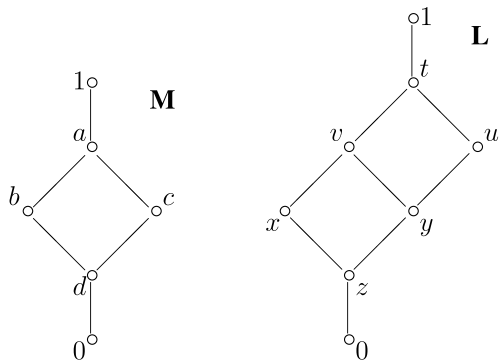

Fig. 3. Hasse diagrams of lattices M and L.

It is worth noting that if M is an (r,s)-sublattice of L then there is a retraction r from L onto M but r needs not be a lower or an upper retraction. Nevertheless, as shown in Remark 2.6 above, there may be more than one retraction from L onto M with the same pseudo-inverse.

Definition 2.7. Let M be a (r_1,s)-sublattice of L. We say that

1. M is a lower (r_1,s)-sublattice of L if r_1 is a lower retraction. Notation: $M < L$ with respect to (r_1,s);
2. M is an upper (r_1,s)-sublattice of L if r_1 is an upper retraction. Notation: $M > L$ with respect to (r_1,s);

3. If r_1 is a lower retraction and there is an upper retraction $r_2 : L \to M$ such that its pseudo-inverse is also s, then M is called a full (r_1, r_2, s)-sublattice of L. Notation: $M \trianglelefteq L$ with respect to (r_1, r_2, s).

Remark 2.7. Let L be a complete bounded lattice. If M is a complete and lower (respectively upper) (r, s)-sublattice of L we write $M \lessdot L$ $(M \gtrdot L)$.

An immediate consequence of the definition of lower (upper) retraction is that, if $M \trianglelefteq L$ then it follows that $s \circ r_1 \leqslant id_L \leqslant s \circ r_2$.

Proposition 2.3. *(See [25].) Let K, M and L be bounded lattices. If $K \trianglelefteq M \trianglelefteq L$ then $K \trianglelefteq L$.*

Proposition 2.4. *Let $M \lessdot L$ with respect to (r, s). For every nonempty set $A \subseteq M$ it holds that $s(\sup A) = \sup s(A)$. In other words, a pseudo-inverse of a lower retract preserves the supremum element.*

Proof. Recall that $s(A) = \{s(t) \in L \mid t \in A\}$. Putting $a = \sup A$ and $a' = \sup s(A)$ we shall prove that $s(a) = a'$.

On the one hand, for all $t \in s(A)$ there is a $k \in A$ such that $t = s(k)$. Since $k \leqslant_M a$ for each $k \in A$, it follows that $t = s(k) \leqslant_L s(a)$ from the monotonicity of s, i.e. $s(a)$ is an upper bound of $s(A)$ and hence $a' \leqslant_L s(a)$ since $a' = \sup s(A)$.

On the other hand, if we take an arbitrary element $k \in A$ it follows that $s(k) \leqslant_L a'$, so $k = r(s(k)) \leqslant_M r(a')$ for all $k \in A$. This means that $r(a')$ is an upper bound of A. Thus, from the definition of supremum we have that $a \leqslant_M r(a')$ and hence $s(a) \leqslant_L s(r(a')) \leqslant_L a'$ since r is a lower retraction. □

Analogously, we can prove the following.

Proposition 2.5. *Let $M \gtrdot L$ with respect to (r, s). For every nonempty set $A \subseteq M$ it holds that $s(\inf A) = \inf s(A)$, i.e. a pseudo-inverse of an upper retract preserves the infimum element.*

Remark 2.8. It is important to remark that in Proposition 2.4 (Proposition 2.5) it was just used that r is a lower retraction (an upper retraction) to prove inequality $s(\sup A) \leqslant_L \sup s(A)$ $(\inf s(A) \leqslant_L s(\inf A))$. It means that $\sup s(A) \leqslant_L s(\sup A)$ $(s(\inf A) \leqslant_L \inf s(A))$ always holds no matter which kind of retraction r is (i.e., lower, upper or neither).

2.3. T-norms and T-conorms on L

A short formalization for the notion of t-norm and t-conorm on bounded lattices is presented here. Moreover, some results are demonstrated as well.

Definition 2.8. (See [5].) Let L be a bounded lattice. A binary operation $T : L \times L \to L$ is a t-norm if, for all $x, y, z \in L$, it satisfies:

1. $T(x, y) = T(y, x)$ (commutativity);
2. $T(x, T(y, z)) = T(T(x, y), z)$ (associativity);
3. If $x \leqslant_L y$ then $T(x, z) \leqslant_L T(y, z)$, $\forall z \in L$ (monotonicity);
4. $T(x, 1_L) = x$ (boundary condition).

Example 2.4. Let L be a bounded lattice. Thus, the function $T : L \times L \to L$ defined by $T(x, y) = x \wedge_L y$ is a t-norm that generalizes the classical fuzzy minimum t-norm, i.e. $T_M : [0, 1] \times [0, 1] \to [0, 1]$ given by $T_M(x, y) = \min\{x, y\}$ for all $x, y \in [0, 1]$.

Dually, it is possible to define the concept of t-conorm.

Definition 2.9. (See [25].) Let L be a bounded lattice. A binary operation $S : L \times L \to L$ is said be a t-conorm if, for all $x, y, z \in L$, we have:

1. $S(x, y) = S(y, x)$ (commutativity);
2. $S(x, S(y, z)) = S(S(x, y), z)$ (associativity);
3. If $x \leqslant_L y$ then $S(x, z) \leqslant_L S(y, z)$, $\forall z \in L$ (monotonicity);
4. $S(x, 0_L) = x$ (boundary condition).

Notice that $T(x, y) \leqslant_L x$ (or $T(x, y) \leqslant_L y$) and $x \leqslant_L S(x, y)$ (or $y \leqslant_L S(x, y)$) for all $x, y \in L$. In fact, $T(x, y) \leqslant_L x \wedge_L y \leqslant_L x$ and $x \leqslant_L x \vee_L y \leqslant_L S(x, y)$.

Example 2.5. Given an arbitrary bounded lattice L, the function S given by $S(x, y) = x \vee_L y$ for all $x, y \in L$ is a t-conorm on L that generalizes the classical fuzzy maximum t-conorm, i.e. $S_M(x, y) = \max\{x, y\}$ for all $x, y \in [0, 1]$.

Proposition 2.6. *Let ρ be an automorphism on L. A t-conorm $S : L \times L \to L$ satisfies*

$$S(x, y) = 1_L \quad \textit{if and only if} \quad x = 1_L \quad \textit{or} \quad y = 1_L \tag{3}$$

if and only if S^ρ also satisfies it. A t-conorm satisfying (3) is called positive.

Proof. We have that $S^\rho(x, y) = 1_L$ if and only if $\rho^{-1}(S(\rho(x), \rho(y))) = 1_L$ if and only if $S(\rho(x), \rho(y)) = 1_L$ if and only if $\rho(x) = 1_L$ or $\rho(y) = 1_L$ (by (3)) if and only if $x = 1_L$ or $y = 1_L$. □

Similarly, it can be proved the following.

Proposition 2.7. *Let ρ be an automorphism on L. A t-norm $T : L \times L \to L$ satisfies*

$$T(x, y) = 0_L \quad \textit{if and only if} \quad x = 0_L \quad \textit{or} \quad y = 0_L \tag{4}$$

if and only if T^ρ also satisfies it. A t-norm satisfying (4) is called positive.

2.4. Negations on L

There are several approaches to the notion of fuzzy negation in order to have a generalization of the classical one [6,7,11,24,25]. We consider here the lattice version of the definition of negation given by Zadeh in [30].

Definition 2.10. A function $N : L \to L$ is called a fuzzy negation if it satisfies:

(N1) $N(0_L) = 1_L$ and $N(1_L) = 0_L$;
(N2) If $x \leqslant_L y$ then $N(y) \leqslant_L N(x)$, for all $x, y \in L$.

Moreover, the negation N is strong if it also satisfies the involution property, namely

(N3) $N(N(x)) = x$, for all $x \in L$.

In case N satisfies

(N4) $N(x) \in \{0_L, 1_L\}$ if and only if $x = 0_L$ or $x = 1_L$

it is called frontier [7].[4] In addition, every element $x \in L$ such that $N(x) = x$ is said to be an equilibrium point of N.

From the point of view of lattice theory a strong negation corresponds to what is known as involution (see [8]).

[4] A negation N is frontier if it is simultaneously non-vanishing and non-filling (see [3]).

Example 2.6. (See [7].) If L is an arbitrary bounded lattice, then the functions $N_\perp, N_\top : L \to L$ defined by

$$N_\perp(x) = \begin{cases} 1_L, & \text{if } x = 0_L; \\ 0_L, & \text{otherwise,} \end{cases}$$

and

$$N_\top(x) = \begin{cases} 0_L, & \text{if } x = 1_L; \\ 1_L, & \text{otherwise,} \end{cases}$$

for each $x \in L$ are fuzzy negations on L.

Proposition 2.8. *(See [7].) Let $N : L \to L$ be a function, ρ be an automorphism on L and $i \in \{1, 2, 3, 4\}$. Then, N satisfies (Ni) if and only if N^ρ satisfies (Ni). Moreover, the element e of L is an equilibrium point of N, i.e. $N(e) = e$ if and only if $\rho^{-1}(e)$ is an equilibrium point of N^ρ.*

3. Extension of t-norms, t-conorms and fuzzy negations

As introduced in [25] we present in this section a method to extend t-norms, t-conorms and fuzzy negations defined on a sublattice taking into account the notion of (r, s)-sublattice given in Definition 2.5.

3.1. The method of extension of fuzzy connectives

Consider an ordinary sublattice M of a bounded lattice L (i.e. $M \subseteq L$) and a t-norm T defined on M. Since a t-norm is particularly a function it is natural to think if it is possible to extend T from M to L in order to obtain a new t-norm T^E on L.

One of the first published works on this issue was proposed by Saminger-Platz et al. [27]. There, it was proposed a method to extend t-norms, namely, given a t-norm T on a (complete) ordinary sublattice M of a lattice L, the function defined by

$$W^L_{T^M}(x, y) = \begin{cases} x \wedge_L y, & \text{if } 1_L \in \{x, y\}; \\ T^{M\cup\{0_L,1_L\}}(x^*, y^*), & \text{otherwise,} \end{cases} \tag{5}$$

where $x^* = \sup_M\{z \mid z \leqslant_L x, z \in M \cup \{0_L, 1_L\}\}$ and

$$T^{M\cup\{0_L,1_L\}}(x, y) = \begin{cases} x \wedge_L y, & \text{if } 1_L \in \{x, y\}; \\ 0_L, & \text{if } 0_L \in \{x, y\}; \\ T^M(x, y), & \text{otherwise,} \end{cases} \tag{6}$$

is an extension of T from M to L.

Seeking to generalize this t-norm extension method for the relaxed notion of sublattice proposed in Definition 2.5, Palmeira and Bedregal presented in [25,26] other way to extend t-norms, t-conorms and fuzzy negations that we present in the following propositions.

Proposition 3.1. *(See [25, Theorem 3.1].) Let $M < L$ with respect to (r, s). If T is a t-norm on M then $T^E : L \times L \to L$ defined by*

$$T^E(x, y) = \begin{cases} x \wedge_L y, & \textit{if } 1_L \in \{x, y\}, \\ s(T(r(x), r(y))), & \textit{otherwise,} \end{cases} \tag{7}$$

is a t-norm which extends T from M to L.

Similarly, it is possible to apply the method of extension of t-norms for t-conorms and fuzzy negations as one can see in Propositions 3.2 and 3.3 below.

Proposition 3.2. *(See [25, Proposition 4.1].) Let $M > L$ with respect to (r, s). If S is a t-conorm on M then S^E : $L \times L \to L$ defined by*

$$S^E(x,y)=\begin{cases} x\vee_L y, & \text{if } 0_L\in\{x,y\},\\ s(S(r(x),r(y))), & \text{otherwise,}\end{cases} \tag{8}$$

is a t-conorm which extends S from M to L.

Proposition 3.3. *(See [25, Proposition 4.2].) Let M be an (r,s)-sublattice of L and $N: M\to M$ be a fuzzy negation. Then $N^E(x)=s(N(r(x)))$ for each $x\in L$ is a fuzzy negation that extends N from M to L.*

It is worth noting that the hypotheses of Proposition 3.3 require only that r should be a retraction (it needs not be neither a lower nor an upper retraction) and hence if r is a lower retraction or an upper retraction the result will remain valid. This fact allows us to extend fuzzy negations in a more flexible way than t-norms and t-conorms.

Remark 3.1. Note that the extension of a fuzzy negation does not preserve involutivity, i.e., in general N^E is not a strong negation even N is. For instance, let $M=[0,1]$ and $L=[0,2]$. Considering the retraction $r: L\to M$ such that $r(x)=x$ if $x\in[0,1]$ and $r(x)=1$ else which has as a pseudo-inverse the function $s: M\to L$ given by $s(x)=x$ for all $x\in[0,1)$ and $s(1)=2$, then if N is an arbitrary strong negation on M we have that $N^E(N^E(4\backslash 3))=2$. This shows that N^E is not involutive.

Proposition 3.4. *Let M be an (r,s)-sublattice of L and $N: M\to M$ a fuzzy negation. Then $e\in M$ is an equilibrium point of N if and only if $s(e)$ is an equilibrium point of N^E.*

Proof. Let $e\in M$ be an equilibrium point of N, i.e. $N(e)=e$. Hence by Proposition 3.3 we have that $N^E(s(e))=s(N(r(s(e))))=s(N(e))=s(e)$, so $s(e)$ is an equilibrium point of N^E.

Conversely, if $s(e)$ is an equilibrium point of N^E, since $N(r(x))=r(N^E(x))$ for each $x\in L$ it follows that $N(e)=N(r(s(e)))=r(N^E(s(e)))=r(s(e))=e$. □

3.2. Extensions and automorphisms

Let M be an (r,s)-sublattice of L. Given an automorphism ρ on M some conditions are also established in [25] conditions under which an automorphism ψ on L defined from ρ is such that $(T^\rho)^E\leqslant(T^E)^\psi$. That is, it is shown the relation between the extension of the conjugate of T and the conjugate of the extension of T.

Proposition 3.5. *(See [25, Theorem 6.1].) Let M be an (r,s)-sublattice of L, ρ be an automorphism on M and T be a t-norm on M. Moreover, suppose $\psi: L\to L$ given by $\psi=s\circ\rho\circ r$ is an automorphism on L. Then $(T^\rho)^E\leqslant(T^E)^\psi$.*

Remark 3.2. Note that under similar conditions it is possible to prove a version of Proposition 3.5 for a given t-conorm S on M. In this case, $r: L\to M$ must be an upper retraction and we have that $(S^E)^\psi\leqslant(S^\rho)^E$.

Proposition 3.6. *Let M be an (r,s)-sublattice of L, ρ be an automorphism on M and N be a fuzzy negation on M. If $\psi: L\to L$ is an automorphism on L such that $r\circ\psi=\rho\circ r$ and $\psi^{-1}\circ s=s\circ\rho^{-1}$ then $(N^\rho)^E=(N^E)^\psi$.*

Proof. By definition, for all $x\in L$ we have

$$\left(N^\rho\right)^E(x)=s\left(\rho^{-1}\left(N\left(\rho\left(r(x)\right)\right)\right)\right) \tag{9}$$

and

$$\left(N^E\right)^\psi(x)=\psi^{-1}\left(s\left(N\left(r\left(\psi(x)\right)\right)\right)\right). \tag{10}$$

By hypothesis $\psi^{-1}\circ s=s\circ\rho^{-1}$ and $r\circ\psi=\rho\circ r$. So by (9) and (10) it follows that

$$\left(N^\rho\right)^E(x)=\psi^{-1}\left(s\left(N\left(r\left(\psi(x)\right)\right)\right)\right)=\left(N^E\right)^\psi(x). \qquad \square$$

Corollary 3.1. *Under the same conditions of Proposition 3.6 if e is an equilibrium point of a strong negation N then $\rho^{-1}\circ\rho^{-1}(e)$ is also an equilibrium point of N.*

Proof. If e is an equilibrium point of N then by Proposition 2.8 $\rho^{-1}(e)$ is an equilibrium point of N^ρ and hence $s \circ \rho^{-1}(e)$ is an equilibrium point of $(N^\rho)^E$ by Proposition 3.4. But by Proposition 3.6 we have that $(N^\rho)^E = (N^E)^\psi$ which allows us to conclude that $s \circ \rho^{-1}(e)$ is an equilibrium point of $(N^E)^\psi$. Applying Proposition 2.8 again it follows that $\psi^{-1} \circ s \circ \rho^{-1}(e)$ is an equilibrium point of N^E, i.e. $N^E(\psi^{-1} \circ s \circ \rho^{-1}(e)) = \psi^{-1} \circ s \circ \rho^{-1}(e)$. This implies that $N(r(\psi^{-1} \circ s \circ \rho^{-1}(e))) = r(\psi^{-1} \circ s \circ \rho^{-1}(e))$ since $N(r(x)) = r(N^E(x))$ for all $x \in M$. In other words, $r \circ \psi^{-1} \circ s \circ \rho^{-1}(e)$ is an equilibrium point of N. But, note that $r \circ \psi^{-1} \circ s = \rho^{-1}$ and hence $\rho^{-1} \circ \rho^{-1}(e)$ is an equilibrium point of N. $\square$

4. Extension of fuzzy implications

This section is devoted to the application of the method of extension of t-norms, t-conorms and fuzzy negations (introduced in the previous section) for implications. We start recalling the notion of fuzzy implication on L and the main properties of this operator.

4.1. Implications on L

There are several ways to define fuzzy implication as one can see in the literature [2,10,16,23,29]. We consider here the notion of implication proposed in [16] in the context of bounded lattices.

Definition 4.1. A function $I : L \times L \to L$ is a fuzzy implication on L if for each $x, y, z \in L$ the following properties hold:

1. First place antitonicity (FPA): if $x \leqslant_L y$ then $I(y, z) \leqslant_L I(x, z)$;
2. Second place isotonicity (SPI): if $y \leqslant_L z$ then $I(x, y) \leqslant_L I(x, z)$;
3. Corner condition 1 (CC1): $I(0_L, 0_L) = 1_L$;
4. Corner condition 2 (CC2): $I(1_L, 1_L) = 1_L$;
5. Corner condition 3 (CC3): $I(1_L, 0_L) = 0_L$.

Example 4.1. Let L be a bounded lattice. Thus the functions $I_\perp, I_\top : L \times L \to L$ given by

$$I_\perp(x, y) = \begin{cases} 1_L, & \text{if } x = 0_L \text{ or } y = 1_L; \\ 0_L, & \text{otherwise,} \end{cases}$$

and

$$I_\top(x, y) = \begin{cases} 0_L, & \text{if } x = 1_L \text{ and } y = 0_L; \\ 1_L, & \text{otherwise,} \end{cases}$$

for all $x, y \in L$ are fuzzy implications.

Consider also the following properties of an implication I on L:

(LB) $I(0_L, y) = 1_L$, for all $y \in L$;
(RB) $I(x, 1_L) = 1_L$, for all $x \in L$;
(CC4) $I(0_L, 1_L) = 1_L$;
(NP) $I(1_L, y) = y$ for each $y \in L$ (left neutrality principle);
(EP) $I(x, I(y, z)) = I(y, I(x, z))$ for all $x, y, z \in L$ (exchange principle);
(IP) $I(x, x) = 1_L$ for each $x \in L$ (identity principle);
(OP) $I(x, y) = 1_L$ if and only if $x \leqslant_L y$ (ordering property);
(IBL) $I(x, I(x, y)) = I(x, y)$ for all $x, y, z \in L$ (iterative Boolean law);
(CP) $I(x, y) = I(N(y), N(x))$ for each $x, y \in L$ with N a fuzzy negation on L (law of contraposition);
(L-CP) $I(N(x), y) = I(N(y), x)$ (law of left contraposition);
(R-CP) $I(x, N(y)) = I(y, N(x))$ (law of right contraposition);
(P) $I(x, y) = 0_L$ if and only if $x = 1_L$ and $y = 0_L$ (positivity).

Notice that (LB) and (RB) together imply property (CC4).

4.2. The extension method for implications

In the following theorem it is presented a way to extend fuzzy implications as an application of the method of extending fuzzy operators introduced in [25].

Theorem 4.1. *Let M be an (r,s)-sublattice of L. If I is an implication on M then the function $I^E : L \times L \to L$ given by*

$$I^E(x,y) = s\big(I\big(r(x), r(y)\big)\big) \tag{11}$$

for all $x, y \in L$, is an implication on L. In this case, I^E is called the extension of I from M to L.

Proof. Let $x, y, z \in L$. We shall prove that I^E satisfies the axioms of Definition 4.1. Thus,

(1) (FPA) Suppose $x \leqslant_L y$. Thus, by monotonicity of r, we have $r(x) \leqslant_L r(y)$ and hence by (FPA) and the isotonicity of s it follows that

$$\begin{aligned} I^E(y,z) &= s\big(I\big(r(y), r(z)\big)\big) \\ &\leqslant_L s\big(I\big(r(x), r(z)\big)\big) \\ &= I^E(x,z). \end{aligned}$$

(2) (SPI) Now, let $y \leqslant_L z$. Again, by monotonicity of r we have $r(y) \leqslant_L r(z)$. Then

$$\begin{aligned} I^E(x,y) &= s\big(I\big(r(x), r(y)\big)\big) \\ &\leqslant_L s\big(I\big(r(x), r(z)\big)\big) \\ &= I^E(x,z). \end{aligned}$$

(3) Moreover, we have that
- (CC1) $I^E(0_L, 0_L) = s(I(r(0_L), r(0_L))) = s(I(0_M, 0_M)) = s(1_M) = 1_L$;
- (CC2) $I^E(1_L, 1_L) = s(I(r(1_L), r(1_L))) = s(I(1_M, 1_M)) = s(1_M) = 1_L$;
- (CC3) $I^E(1_L, 0_L) = s(I(r(1_L), r(0_L))) = s(I(1_M, 0_M)) = s(0_M) = 0_L$.

Therefore, by (1), (2) and (3) it can be concluded that I^E is an implication on L. □

It is worth noting that, as it happened for fuzzy negations, we do not need to impose any additional property to the retract r for extending implications. Note that for t-conorms, r must be an upper retraction (for t-norms r must be a lower retraction, see [25]).

Proposition 4.1. *Under the same conditions as in Theorem 4.1, if I is an implication on M satisfying any property* (LB), (RB), (CC4), (EP), (IP), (IBL), (CP), (L-CP) *and* (R-CP) *then I^E is an implication on L which satisfies the same property.*

Proof. If I is an implication on M then, by Theorem 4.1, $I^E(x,y) = s(I(r(x), r(y)))$ for all $x, y \in L$. Thus:

(LB) By hypothesis $I(0_M, x) = 1_M$ for all $x \in M$. Then $I^E(0_L, y) = s(I(r(0_L), r(y))) = s(I(0_M, r(y))) = s(1_M) = 1_L$ for all $y \in L$.

(RB) Now, considering that $I(x, 1_M) = 1_M$ for all $x \in M$, it follows that $I^E(a, 1_L) = s(I(r(a), r(1_L))) = s(I(r(a), 1_M)) = s(1_M) = 1_L$ for all $a \in L$.

(CC4) $I^E(0_L, 1_L) = s(I(r(0_L), r(1_L))) = s(I(0_M, 1_M)) = s(1_M) = 1_L$.

(EP)

$$\begin{aligned}
I^E\big(x, I^E(y,z)\big) &= s\Big(I\big(r(x), r\big(I^E(y,z)\big)\big)\Big) && \text{by Eq. (11)}\\
&= s\Big(I\Big(r(x), r\Big(s\big(I\big(r(y), r(z)\big)\big)\Big)\Big)\Big) && \text{by Eq. (11)}\\
&= s\Big(I\big(r(x), I\big(r(y), r(z)\big)\big)\Big) && \text{by Def. 2.4}\\
&= s\Big(I\big(r(y), I\big(r(x), r(z)\big)\big)\Big) && \text{by (EP)}\\
&= s\Big(I\Big(r(y), r\Big(s\big(I\big(r(x), r(z)\big)\big)\Big)\Big)\Big) && \text{by Def. 2.4}\\
&= I^E\big(y, I^E(x,z)\big)
\end{aligned}$$

(IP) If $I(x,x) = 1_M$ for each $x \in M$ then $I^E(y,y) = s(I(r(y), r(y))) = s(1_M) = 1_L$ for all $y \in L$.
(IBL) For all $x, y \in L$, if I satisfies (IBL) we have that

$$\begin{aligned}
I^E\big(x, I^E(x,y)\big) &= s\Big(I\big(r(x), r\big(I^E(x,y)\big)\big)\Big) && \text{by Eq. (11)}\\
&= s\Big(I\Big(r(x), r\Big(s\big(I\big(r(x), r(y)\big)\big)\Big)\Big)\Big) && \text{by Eq. (11)}\\
&= s\Big(I\big(r(x), I\big(r(x), r(y)\big)\big)\Big) && \text{by Def. 2.4}\\
&= s\big(I\big(r(x), r(y)\big)\big) && \text{by (IBL)}\\
&= I^E(x,y)
\end{aligned}$$

(CP) Let N be a fuzzy negation on M such that $I(x,y) = I(N(y), N(x))$. Thus, by Proposition 3.3, N^E is a fuzzy negation on L. Hence

$$\begin{aligned}
I^E\big(N^E(y), N^E(x)\big) &= s\Big(I\big(r\big(N^E(y)\big), r\big(N^E(x)\big)\big)\Big) && \text{by Eq. (11)}\\
&= s\Big(I\Big(r\Big(s\big(N\big(r(y)\big)\big)\Big), r\Big(s\big(N\big(r(x)\big)\big)\Big)\Big)\Big) && \text{by Prop. 3.3}\\
&= s\Big(I\big(N\big(r(y)\big), N\big(r(x)\big)\big)\Big) && \text{by Def. 2.4}\\
&= s\big(I\big(r(x), r(y)\big)\big) && \text{by (CP)}\\
&= I^E(x,y)
\end{aligned}$$

(L-CP) Now, if N is a fuzzy negation for which I satisfies (L-CP), then

$$\begin{aligned}
I^E\big(N^E(x), y\big) &= s\Big(I\big(r\big(N^E(x)\big), r(y)\big)\Big) && \text{by Eq. (11)}\\
&= s\Big(I\Big(r\Big(s\big(N\big(r(x)\big)\big)\Big), r(y)\Big)\Big) && \text{by Prop. 3.3}\\
&= s\Big(I\big(N\big(r(x)\big), r(y)\big)\Big) && \text{by Def. 2.4}\\
&= s\Big(I\big(N\big(r(y)\big), r(x)\big)\Big) && \text{by (L-CP)}\\
&= s\Big(I\Big(r\Big(s\big(N\big(r(y)\big)\big)\Big), r(x)\Big)\Big) && \text{by Def. 2.4}\\
&= I^E\big(N^E(y), x\big)
\end{aligned}$$

(R-CP) Analogously to previous item. □

Example 4.2. We show here that the method of extending implications fails in preserving properties (NP) and (OP). Let M and L be the bounded lattices shown in Fig. 3. It is clear that the function I given by

$$I(x,y) = \begin{cases} 1_M, & \text{if } x \leqslant_M y; \\ y, & \text{otherwise,} \end{cases}$$

is a fuzzy implication on M (this function is a generalization of Gödel implication for an arbitrary bounded lattice). It can be easily seen in Table 3 that implication I satisfies the properties (NP) and (OP).

Table 3
Implication on M.

I	0_M	d	c	b	a	1_M
0_M	1_M	1_M	1_M	1_M	1_M	1_M
d	0_M	1_M	1_M	1_M	1_M	1_M
c	0_M	d	1_M	b	1_M	1_M
b	0_M	d	c	1_M	1_M	1_M
a	0_M	d	c	b	1_M	1_M
1_M	0_M	d	c	b	a	1_M

However, considering the lower retraction $r(x) = \sup\{z \in M \mid s(z) \leqslant_L x\}$ from L into M and its pseudo-inverse s defined by $s(1_M) = 1_L$, $s(a) = v$, $s(b) = x$, $s(c) = y$, $s(d) = z$ and $s(0_M) = 0_L$ (see Example 2.3), the extension I^E of I does not satisfy properties (NP) and (OP). Indeed, if we take the pair $(1_L, t) \in L^2$ then we have

$$I^E(1_L, t) = s\big(I\big(r(1_L), r(t)\big)\big) = s\big(I(1_M, a)\big) = s(a) = v \neq t$$

i.e. for I^E the property (NP) does not hold. To see that I^E does not satisfy (OP) it is enough to take the pair $(u, v) \in L^2$ and consider that

$$I^E(u, v) = s\big(I\big(r(u), r(v)\big)\big) = s\big(I(c, a)\big) = s(1_M) = 1_L$$

while we have that $u \parallel v$. In other words, $I^E(m, n) = 1_L$ does not imply that $m \leqslant_L n$, for all $m, n \in L$.

However, if I is a fuzzy implication on M satisfying (NP) and (OP), the following weak versions of these properties hold for its extension I^E:

(W-NP): For each $y \in L$, if I satisfies (NP), we have that

$$I^E(1_L, y) = s\big(I\big(r(1_L), r(y)\big)\big) = s\big(I\big(1_M, r(y)\big)\big) = s\big(r(y)\big) \leqslant_L y.$$

(L-OP): If $x \leqslant_L y$ then $I^E(x, y) = 1_L$. Indeed, let $x, y \in L$ such that $x \leqslant_L y$ and suppose that I satisfies (OP). In this case, $r(x) \leqslant_M r(y)$ since r is monotone. Thus, by (OP), $I(r(x), r(y)) = 1_M$ and hence $I^E(x, y) = s(I(r(x), r(y))) = s(1_M) = 1_L$.

Remark 4.1. It is worth noting also that property (P) is not preserved by this method of extending fuzzy implications, in general. It is easy to see that if an implication I satisfies (P) then we have that $I^E(1_L, 0_L) = 0_L$. But if $I^E(x, y) = 0_L$ it does not imply that $x = 1_L$ and $y = 0_L$. For instance, let M and L_1 be the bounded lattices shown in Fig. 2 and take the retraction $r_1 : L_1 \to M$ with pseudo-inverse $s_1 : M \to L_1$ as defined in Example 2.2. In this case the fuzzy implication $I_\top : M \times M \to M$ given by $I_\top(x, y) = a$ if $x = d$ and $y = a$ and $I_\top(x, y) = d$ else (see Example 4.1) satisfies (P) but for its extension $(I_\top)^E$ we have that

$$(I_\top)^E(5, 2) = s_1\big(I_\top\big(r_1(5), r_1(2)\big)\big) = s_1\big(I_\top(d, a)\big) = s_1(a) = 1$$

which means that there are $x, y \in L_1$ (in this case $x = 5$ and $y = 2$. Notice that $0_{L_1} = 1$ and $1_{L_1} = 5$) such that $(I_\top)^E(x, y) = 0_{L_1}$ but $(x, y) \neq (1_{L_1}, 0_{L_1})$. The following proposition shows a weak version of the sufficiency part of property (P).

Proposition 4.2. *Let M be an (r, s)-sublattice of L and I be a fuzzy implication on M which satisfies property* (P). *Then*:

1. *If $M < L$ and $I^E(x, y) = 0_L$ then $x = 1_L$;*
2. *If $M > L$ and $I^E(x, y) = 0_L$ then $y = 0_L$.*

Proof. If M is an (r, s)-sublattice of L then there exists a retraction $r : L \to M$ and a pseudo-inverse $s : M \to L$ such that $r \circ s = id_M$. Thus, supposing that $I^E(x, y) = 0_L$ for some $x, y \in L$, we get $s(I(r(x), r(y))) = 0_L = s(0_M)$, and hence by injectivity of s and (P), we have that

$$I\big(r(x), r(y)\big) = 0_M \quad \text{implies} \quad r(x) = 1_M \quad \text{and} \quad r(y) = 0_M \tag{12}$$

1. If $M < L$ then we also have that $s \circ r \leqslant id_L$ and hence $1_L = s(1_M) = s(r(x)) \leqslant_L x$ by (12). Therefore, we must have $x = 1_L$ since 1_L is the supremum element of L.
2. If $M > L$ it follows that $id_L \leqslant s \circ r$. Again, by (12) we have $y \leqslant_L s(r(y)) = s(0_M) = 0_L$. It means that $y = 0_L$ considering that 0_L is the infimum of L. $\square$

4.3. Implications and automorphisms

Now, we turn our attention to develop some results involving extensions and conjugates (as defined in (2)) of a given fuzzy implication I.

Proposition 4.3. *Let ρ be an automorphism on L, $I : L \times L \to L$ be a function and $Q \in \{$(FPA), (SPI), (CC1), (CC2), (CC4), (LB), (RB)$\}$. I satisfies Q if and only if I^ρ also satisfies Q.*

Proof. See Proposition 10 in [7]. $\square$

Corollary 4.1. *Let M be an (r, s)-sublattice of L, ρ be an automorphism on M and $I : M \times M \to M$ be a function. If I satisfies $Q \in \{$(FPA), (SPI), (CC1), (CC2), (CC4), (LB), (RB)$\}$ then $(I^\rho)^E$ also satisfies Q.*

Proof. Straightforward from Proposition 4.1 and Proposition 4.3. $\square$

Consider the following issue: Let M be an (r, s)-sublattice of L, I an implication on M and $\rho : M \to M$ an automorphism. Is it possible to define an automorphism ψ on L from ρ in order to have $(I^\rho)^E = (I^E)^\psi$?

A natural candidate to solve this problem would be ρ^E, i.e. $\rho^E(x) = s(\rho(r(x)))$ for each $x \in L$. But ρ^E is not an automorphism in general. For instance, let M and L_1 be the bounded lattices shown in Fig. 2 and $r_1 : L_1 \to M$ the lower retract defined in Example 2.2 which has as pseudo-inverse the homomorphism $s_1 : L_1 \to M$ given by $s_1(a) = 1$, $s_1(b) = 2$, $s_1(c) = 3$ and $s_1(d) = 5$ (as in Example 2.2, too). Thus, if we define the automorphism $\rho : M \to M$ by $\rho(a) = a$, $\rho(b) = c$, $\rho(c) = b$ and $\rho(d) = d$ then it is easy to see that $\rho^E(x) = s(\rho(r(x)))$ for each $x \in L_1$, is not an automorphism since $\rho^E(1) = \rho^E(2) = 1$, i.e. ρ^E is not injective.

Theorem 4.2. *Let M be an (r, s)-sublattice of L, ρ be an automorphism on M and I be an implication on M. Moreover, suppose $\psi : L \to L$ is an automorphism on L such that $r \circ \psi = \rho \circ r$ and $\psi^{-1} \circ s = s \circ \rho^{-1}$. Then $(I^\rho)^E = (I^E)^\psi$.*

Proof. By definition, for all $x, y \in L$ we have

$$\big(I^\rho\big)^E(x, y) = s\big(\rho^{-1}\big(I\big(\rho\big(r(x)\big), \rho\big(r(y)\big)\big)\big)\big) \tag{13}$$

and

$$\big(I^E\big)^\psi(x, y) = \psi^{-1}\big(s\big(I\big(r\big(\psi(x)\big), r\big(\psi(y)\big)\big)\big)\big). \tag{14}$$

Considering that $\psi^{-1} \circ s = s \circ \rho^{-1}$ and $\psi \circ s = s \circ \rho$ (since $\psi = s \circ \rho \circ r$ and s is a pseudo-inverse of r) then by (13) and (14) it follows that

$$\big(I^\rho\big)^E(x, y) = \psi^{-1}\big(s\big(I\big(r\big(\psi(x)\big), r\big(\psi(y)\big)\big)\big)\big) = \big(I^E\big)^\psi(x, y). \qquad \square$$

4.4. Relationship between extensions of negations and implications

There exists a natural way to define a particular class of fuzzy negations from implications on [0, 1] based on the fact that a propositional formula p is logically equivalent (in classical logic) to $p \to \bot$ where $\bot$ denotes the absurd (see Lemma 1.4.14, p. 18 in [2]). In [7] Bedregal et al. have generalized this concept for bounded lattices as follows.

Proposition 4.4. *Let L be a bounded lattice. If a function $I : L \times L \to L$ satisfies* (FPA), (CC1) *and* (CC3) *then the function $N_I : L \to L$ defined for each $x \in L$ by*

$$N_I(x) = I(x, 0_L) \tag{15}$$

is a fuzzy negation on L called the natural negation of I.

The next lemma provides necessary conditions for N_I to be a fuzzy negation and also establishes some properties.

Lemma 4.1. *(See [7].) Let L be a bounded lattice. If a function $I : L \times L \to L$ satisfies* (EP) *and* (OP) *then*

1. N_I *is a fuzzy negation*;
2. $x \leqslant_L N_I(N_I(x))$ *for each* $x \in L$;
3. $N_I \circ N_I \circ N_I = N_I$.

Proposition 4.5. *Let M be an (r, s)-sublattice of L and $I : M \times M \to M$ be a function satisfying* (EP) *and* (OP). *Then*

1. $N_{I^E}(x) := I^E(x, 0_L)$ *for all $x \in L$ is a fuzzy negation on L*;
2. $N_{I^E} = (N_I)^E$;
3. *If r is an upper retraction then $x \leqslant_L N_{I^E}(N_{I^E}(x))$ for each $x \in L$*;
4. $N_{I^E} \circ N_{I^E} \circ N_{I^E} = N_{I^E}$.

Proof.

1. Let $x, y \in L$ such that $x \leqslant_L y$. Since r is monotone then $r(x) \leqslant_M r(y)$ and hence $I(r(y), 0_M) \leqslant_M I(r(x), 0_M)$ by (FPA) (properties (EP) and (OP) imply (FPA) as one can see in Lemma 6 in [2] for $\langle[0, 1], \leqslant\rangle$. The proof for a generic lattice L is similar). Thus

$$N_{I^E}(y) = s\big(I\big(r(y), r(0_L)\big)\big) = s\big(I\big(r(y), 0_M\big)\big) \leqslant_M s\big(I\big(r(x), 0_M\big)\big) = N_{I^E}(x).$$

Moreover,

$$N_{I^E}(0_L) = s\big(I\big(r(0_L), r(0_L)\big)\big) = s\big(I(0_M, 0_M)\big) = s(1_M) = 1_L$$

and

$$N_{I^E}(1_L) = s\big(I\big(r(1_L), r(0_L)\big)\big) = s\big(I(1_M, 0_M)\big) = s(0_M) = 0_L.$$

Therefore, it can be concluded that N_{I^E} is a fuzzy negation on L.

2. For each $x \in L$ we have

$$N_{I^E}(x) = s\big(I\big(r(x), r(0_L)\big)\big) = s\big(I\big(r(x), 0_M\big)\big) = s\big(N_I\big(r(x)\big)\big) = (N_I)^E(x).$$

3. Note that for each $x \in L$

$$\begin{aligned} N_{I^E}\big(N_{I^E}(x)\big) &= s\big(I\big(r\big(N_{I^E}(x)\big), 0_M\big)\big) \\ &= s\big(I\big(r\big(s\big(I\big(r(x), 0_M\big)\big)\big), 0_M\big)\big) \\ &= s\big(I\big(I\big(r(x), 0_M\big), 0_M\big)\big) \\ &= s\big(N_I\big(N_I\big(r(x)\big)\big)\big). \end{aligned} \tag{16}$$

If r is an upper retraction then $id_L \leqslant s \circ r$. By item 2 of Lemma 4.1 we have that $r(x) \leqslant_L N_I(N_I(r(x)))$ for each $x \in L$ and hence $x \leqslant_L s(r(x)) \leqslant_L s(N_I(N_I(r(x))))$. Thus by (16) it follows that $x \leqslant_L N_{I^E}(N_{I^E}(x))$ for each $x \in L$.

4. Recall that $r \circ s = id_M$ and that $N_I \circ N_I \circ N_I = N_I$ (by item 3, Lemma 4.1). Thus,

$$\begin{aligned} N_{I^E}\big(N_{I^E}(N_{I^E}(x))\big) &= s\big(I\big(r\big(N_{I^E}\big(N_{I^E}(x)\big)\big), 0_M\big)\big) \\ &= s\big(I\big(r\big(s\big(I\big(r\big(s\big(I\big(r(x), 0_M\big)\big)\big), 0_M\big)\big)\big), 0_M\big)\big) \\ &= s\big(I\big(I\big(I\big(r(x), 0_M\big), 0_M\big), 0_M\big)\big) \\ &= s\big(N_I\big(N_I\big(N_I\big(r(x)\big)\big)\big)\big) \\ &= s\big(N_I\big(r(x)\big)\big) \\ &= s\big(N_I\big(r(x), 0_M\big)\big) \\ &= N_{I^E}(x) \end{aligned}$$

for each $x \in L$. □

5. Extension of (S, N)-implication

A special type of fuzzy implication that we would like to study is the (S, N)-implication, that is, an implication defined from a t-conorm S and a fuzzy negation N.

Definition 5.1. Let S be a t-conorm on L and N be a fuzzy negation on L. The function $I_{S,N} : L \times L \to L$ given by

$$I_{S,N}(x, y) = S\big(N(x), y\big) \tag{17}$$

for all $x, y \in L$ is called an (S, N)-implication. If N is strong then I is called a strong implication or S-implication. In this case, S and N are said to be the generators of I.

Proposition 5.1. *Let $M > L$ with respect to (r, s). If S is a t-conorm on M and N is a negation on M then a function $I_{S^E,N^E} : L \times L \to L$ defined by $I_{S^E,N^E}(x, y) = S^E(N^E(x), y)$ for all $x, y \in L$ is an implication in the sense of Definition* 4.1.

Proof. Straightforward from Propositions 3.2 and 3.3. □

Corollary 5.1. *Under the same conditions as in Proposition* 5.1 *it follows that I is an (S, N)-implication on M generated from S and N if and only if I^E is an (S, N)-implication on L generated from S^E and N^E.*

It is important to point out here that in general if I is an S-implication then I^E is not an S-implication since the extension of a strong negation needs not be a strong negation (see Remark 3.1).

Proposition 5.2. *Let $M > L$ with respect to (r, s). Considering S a t-conorm and N a negation both defined on M then it follows that $I_{S^E,N^E} \leqslant (I_{S,N})^E$.*

Proof. By Proposition 5.1 we have that

$$I_{S^E,N^E}(x, y) = S^E\big(N^E(x), y\big) = \begin{cases} N^E(x) \vee_L y, & 0_L \in \{N^E(x), y\} \\ s(S(r(N^E(x)), r(y))), & \text{otherwise.} \end{cases}$$

Since $N^E(x) = s(N(r(x)))$ and $r \circ s = id_M$ then

$$I_{S^E,N^E}(x, y) = \begin{cases} N^E(x) \vee_L y, & 0_L \in \{N^E(x), y\}, \\ s(S(N(r(x))), r(y)), & \text{otherwise.} \end{cases}$$

On the other hand, $(I_{S,N})^E(x, y) = s(I_{S,N}(r(x), r(y))) = s(S(N(r(x)), r(y)))$ for all $x, y \in L$. Thus, it is clear that

$$I_{S^E,N^E}(x, y) = (I_{S,N})^E(x, y) \quad \text{whenever } 0_L \notin \big\{N^E(x), y\big\}. \tag{18}$$

Moreover, if $y = 0_L$ then

$$\begin{aligned} I_{S^E,N^E}(x,0_L) &= N^E(x) = s\big(N\big(r(x)\big)\big) \\ &= s\big(S\big(N\big(r(x)\big),0_M\big)\big) \\ &= s\big(I_{S,N}\big(r(x),0_M\big)\big) \\ &= (I_{S,N})^E(x,0_L) \end{aligned} \tag{19}$$

Now, if $N^E(x) = 0_L$ then $s(N(r(x))) = 0_L = s(0_M)$ which implies $N(r(x)) = 0_M$ since s is injective. Hence

$$\begin{aligned} I_{S^E,N^E}(x,y) &= y \\ &\leqslant_L s\big(r(y)\big) \\ &= s\big(S\big(0_M, r(y)\big)\big) \\ &= s\big(S\big(N\big(r(x)\big), r(y)\big)\big) \\ &= s\big(I_{S,N}\big(r(x), r(y)\big)\big) \\ &= (I_{S,N})^E(x,y) \end{aligned} \tag{20}$$

Therefore, by (18), (19) and (20) it can be concluded that $I_{S^E,N^E}(x,y) \leqslant_L (I_{S,N})^E(x,y)$ for all $x, y \in L$. □

Corollary 5.2. *Under the same conditions as in Proposition 5.2 it follows that* $N_{I_{S^E,N^E}} \leqslant_L N_{(I_{S,N})^E}$.

Proof. Direct from Proposition 4.5 and Proposition 5.2. □

Proposition 5.3. *(See [7].) Let $I : L \times L \to L$ be a function and ρ an automorphism on L. Thus, I is an (S,N)-implication on L generated from S and N if and only if I^ρ is an (S,N)-implication on L generated from S^ρ and N^ρ. In other words, $(I_{S,N})^\rho = I_{S^\rho,N^\rho}$.*

Proposition 5.4. *Let $M > L$ with respect to (r,s) and ρ, S, N be respectively an automorphism, a t-conorm and a fuzzy negation on M. Moreover, suppose the function $\psi : L \to L$ is an automorphism on L such that $r \circ \psi = \rho \circ r$ and $\psi^{-1} \circ s = s \circ \rho^{-1}$. Then $(I_{S^E,N^E})^\psi \leqslant (I_{S^\rho,N^\rho})^E$.*

Proof. For each $x, y \in L$ it follows that

$$\begin{aligned} \big(I_{S^E,N^E}\big)^\psi(x,y) &= I_{(S^E)^\psi,(N^E)^\psi}(x,y) && \text{by Prop. 5.3} \\ &= \big(S^E\big)^\psi\big(\big(N^E\big)^\psi(x), y\big) && \text{by Eq. (17)} \\ &\leqslant_L \big(S^\rho\big)^E\big(\big(N^E\big)^\psi(x), y\big) && \text{by Rem. 3.2} \\ &= \big(S^\rho\big)^E\big(\big(N^\rho\big)^E(x), y\big) && \text{by Prop. 3.6} \\ &= I_{(S^\rho)^E,(N^\rho)^E}(x,y) && \text{by Eq. (17)} \\ &\leqslant_L (I_{S^\rho,N^\rho})^E(x,y) && \text{by Prop. 5.2} \end{aligned}$$ □

Corollary 5.3. *Under the same conditions as in Proposition 5.4 it follows that* $N_{(I_{S^E,N^E})^\psi} \leqslant N_{(I_{S^\rho,N^\rho})^E}$.

6. Extension of R-implications

Taking into account that there exists an isomorphism between classical two-valued logic and classical set theory, it is possible to see that if K and G are subsets of a set X then the identity

$$K^c \cup G = (K \backslash G)^c = \bigcup\{P \subseteq X \mid K \cap P \subseteq G\}$$

holds, where K^c is the complement of set K (see [3]).

The R-implications (residuated implications) are generalizations of this identity in fuzzy logic.

Definition 6.1. Let L be a complete bounded lattice. A function $I : L \times L \to L$ is called an R-implication if there exists a t-norm T such that for all $x, y \in L$ we have

$$I(x, y) = \sup\{t \in L \mid T(x, t) \leqslant_L y\}. \tag{21}$$

We denote this implication generated from a t-norm T by I_T.

Proposition 6.1. *Let $M \lessdot L$ with respect to (r, s). If T is a t-norm on M and N is a negation on M then the function $I_{T^E} : L \times L \to L$ defined by $I_{T^E}(x, y) = \sup\{t \in L \mid T^E(x, t) \leqslant_L y\}$ for all $x, y \in L$ is an implication in the sense of Definition 4.1.*

Proof. Straightforward from Definition 6.1 and the fact that under these hypotheses T^E is a t-norm on L as proved in Proposition 3.1. □

Theorem 6.1. *Let $M \lessdot L$ with respect to (r, s). If T is a t-norm on M then $(I_T)^E \leqslant I_{T^E}$.*

Proof. Firstly, take $x, y \in L$ such that $1_L \notin \{x, y\}$. In this case we have

$$(I_T)^E(x, y) = s\big(I_T\big(r(x), r(y)\big)\big) = s\big(\sup\big\{z \in M \mid T\big(r(x), z\big) \leqslant_M r(y)\big\}\big) \tag{22}$$

and

$$\begin{aligned}(I_{T^E})(x, y) &= \sup\big\{t \in L \mid T^E(x, t) \leqslant_L y\big\} \\ &= \sup\big\{t \in L \mid s\big(T\big(r(x), r(t)\big)\big) \leqslant_L y\big\} \\ &= \sup\big\{t \in L \mid T\big(r(x), r(t)\big) \leqslant_M r(y)\big\}.\end{aligned} \tag{23}$$

If

$$A = \big\{z \in M \mid T\big(r(x), z\big) \leqslant_M r(y)\big\} \quad \text{and} \quad B = \big\{t \in L \mid T\big(r(x), r(t)\big) \leqslant_M r(y)\big\}$$

for all $z \in A$, by (22), we have that $T(r(x), r(s(z))) = T(r(x), z) \leqslant_M r(y)$ which means that $s(z) \in B$ for each $z \in A$, i.e. $s(A) \subseteq B$ and hence $\sup s(A) \leqslant \sup B$ (note that if $C \subseteq D$ then $\sup C \leqslant \sup D$, see [22]). But $s(\sup A) = \sup s(A)$ by Proposition 2.4, so we can conclude that $s(\sup A) \leqslant \sup B$. Therefore $(I_T)^E \leqslant I_{T^E}$.

Finally, to complete the proof we must consider the case $1_L \in \{x, y\}$. We have three possibilities:

(i) $x = y = 1_L$:

$$\begin{aligned}(I_T)^E(1_L, 1_L) &= s\big(I_T(r(1_L), r(1_L))\big) \\ &= s\big(I_T(1_M, 1_M)\big) \\ &= s\big(\sup\big\{z \in M \mid T(1_M, z) \leqslant_M 1_M\big\}\big) \\ &= s\big(\sup\{z \in M \mid z \leqslant_M 1_M\}\big) \\ &= s(\sup M) = s(1_M) = 1_L\end{aligned}$$

and

$$(I_{T^E})(1_L, 1_L) = \sup\big\{t \in L \mid T^E(1_L, t) \leqslant_L 1_L\big\} = \sup L = 1_L.$$

(ii) $x \neq 1_L$ and $y = 1_L$:

$$\begin{aligned}(I_T)^E(x, 1_L) &= s\big(I_T\big(r(x), r(1_L)\big)\big) \\ &= s\big(I_T\big(r(x), 1_M\big)\big) \\ &= s\big(\sup\big\{z \in M \mid T\big(r(x), z\big) \leqslant_M 1_M\big\}\big) \\ &= s(\sup M) = s(1_M) = 1_L\end{aligned}$$

and

$$(I_{T^E})(x, 1_L) = \sup\{t \in L \mid T^E(x,t) \leqslant_L 1_L\} = \sup L = 1_L.$$

(iii) $x = 1_L$ and $y \neq 1_L$:

$$\begin{aligned}(I_T)^E(1_L, y) &= s\big(I_T\big(r(1_L), r(y)\big)\big)\\ &= s\big(I_T\big(1_M, r(y)\big)\big)\\ &= s\big(\sup\{z \in M \mid T(1_M, z) \leqslant_M r(y)\}\big)\\ &= s\big(\sup\{z \in M \mid z \leqslant_M r(y)\}\big)\end{aligned}$$

and

$$(I_{T^E})(1_L, y) = \sup\{t \in L \mid T^E(1_L, t) \leqslant_L y\} = \sup\{t \in L \mid t \leqslant_L y\}.$$

Note that if $z \leqslant_M r(y)$ then $s(z) \leqslant_L s(r(y)) \leqslant_L y$ and hence using the same argumentation of the first part of the proof of this proposition we can conclude that $(I_T)^E(1_L, y) \leqslant_L (I_{T^E})(1_L, y)$.

Therefore by (i), (ii) and (iii) we have that $(I_T)^E(x, y) \leqslant_L (I_{T^E})(x, y)$ when $1_L \in \{x, y\}$. □

Corollary 6.1. *Let $M \lessdot L$ with respect to (r, s). If T is a t-norm on M then $N_{(I_T)^E} \leqslant N_{(I_{T^E})}$. In other words, the natural negation of $(I_T)^E$ is less than or equal to the natural negation of I_{T^E}.*

Proof. Straightforward from Theorem 6.1 and Definition 6.1. □

Proposition 6.2. *Let $M \lessdot L$ with respect to (r, s) and ρ, T be respectively an automorphism and a t-norm on M. Moreover, suppose the function $\psi : L \to L$ is an automorphism such that $r \circ \psi = \rho \circ r$ and $\psi^{-1} \circ s = s \circ \rho^{-1}$. Then $I_{(T^E)^\psi} \leqslant I_{(T^\rho)^E}$.*

Proof. Let $x, y \in L$, $A = \{t \in L \mid (T^E)^\psi(x,t) \leqslant_L y\}$ and $B = \{k \in L \mid (T^\rho)^E(x,k) \leqslant_L y\}$. Thus, for all $t \in A$ we have $(T^E)^\psi(x,t) \leqslant_L y$ and hence $(T^\rho)^E(x,t) \leqslant_L y$ (since $(T^\rho)^E \leqslant (T^E)^\psi$ by Proposition 3.5), i.e. $t \in B$. It means that $A \subseteq B$ and $\sup A \leqslant \sup B$. Therefore,

$$\begin{aligned}I_{(T^E)^\psi}(x, y) &= \sup\{t \in L \mid (T^E)^\psi(x,t) \leqslant_L y\}\\ &\leqslant \sup\{t \in L \mid (T^\rho)^E(x,t) \leqslant_L y\}\\ &= I_{(T^\rho)^E}(x, y) \qquad \square\end{aligned}$$

Corollary 6.2. *Under the same conditions as in Proposition 6.2 it follows that $N_{I_{(T^E)^\psi}} \leqslant N_{I_{(T^\rho)^E}}$.*

7. Final remarks

As shown is this paper, the method of extending fuzzy connectives presented in [25] can be naturally applied for extending lattice-valued fuzzy implications. An advantage of extending implications as well as negations is the possibility of taking any kind of retraction (lower, upper or neither). This is not true when we are dealing with t-norms and t-conorms (see Propositions 3.1 and 3.2). On the other hand, this extension fails in preserving some properties such as (NP) and (OP), which gives rise to the problem of studying weak versions of them.

For future works we are interested in applying the extension method for other particular classes of fuzzy implications such as QL-implications and implications defined from uninorms in the same way we have extended other fuzzy operators as can be seen in [24,26].

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